Chapter 10

Zweier I-Convergent Double Sequence Spaces Defined by Orlicz Function

10.1 Introduction

Recently Vakeel. A. Khan et. al.[37] introduced and studied the following classes of sequence spaces:

$$\mathcal{Z}^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^{/} - L|}{\rho}) = 0 \text{ for some L and } \rho > 0\},$$

$$\mathcal{Z}_0^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^{/}|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

$$\mathcal{Z}_\infty^I(M) = \{(x_k) \in \omega : \sup_k M(\frac{|x_k^{/}|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

Also we denote by

$$m_{\mathcal{Z}}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}^I(M)$$

and

$$m_{\mathcal{Z}_0}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}_0^I(M).$$

10.2 Main Results

In this Chapter we introduce the following classes of Zweier I-Convergent double sequence spaces defined by the Orlicz function.

$${}_2\mathcal{Z}^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|x_{ij}^{'} - L|}{\rho}) = 0$$
$$\text{for some L} \in \mathbb{C} \text{ , and } \rho > 0\},$$

$${}_2\mathcal{Z}_0^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|x_{ij}^{'}|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

"Mazur and Orlicz are direct pupils of Banch; they represent the theory of operations today in poland and their names cover of "Studia Mathematica" indicate direct continuation of Banach's scientific programme."-Hugo Steinhauss

$$
{}_2\mathcal{Z}^I_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : \text{there exist } K > 0 : M(\frac{|x'_{ij}|}{\rho}) \geq K \text{ for some } \rho > 0 \in I\}.
$$

$$
{}_2\mathcal{Z}_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} M(\frac{|x'_{ij}|}{\rho}) < \infty\}
$$

Also we denote by

$$
m^I_{{}_2\mathcal{Z}}(M) = {}_2\mathcal{Z}^I_\infty(M) \cap {}_2\mathcal{Z}^I(M)
$$

and

$$
m^I_{{}_2\mathcal{Z}_0}(M) = {}_2\mathcal{Z}^I_\infty(M) \cap {}_2\mathcal{Z}^I_0(M).
$$

Throughout the chapter, for the sake of convenience, we will denote by $Z^p(x_k) = x'$, $Z^p(y_k) = y'$, $Z^p(z_k) = z'$ for $x, y, z \in \omega$.

Theorem 10.2.1. For any Orlicz function M, the classes of sequences ${}_2\mathcal{Z}^I(M)$, ${}_2\mathcal{Z}^I_0(M)$, ${}_2m^I_{\mathcal{Z}}(M)$ and ${}_2m^I_{\mathcal{Z}_0}(M)$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(M)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I(M)$ and let α, β be scalars. Then there exists positive numbers ρ_1 and ρ_2 such that

$$
I - \lim M(\frac{|x'_{ij} - L_1|}{\rho_1}) = 0, \text{ for some } L_1 \in \mathbb{C} \;;
$$

$$
I - \lim M(\frac{|y'_{ij} - L_2|}{\rho_2}) = 0, \text{ for some } L_2 \in \mathbb{C} \;;
$$

That is for a given $\epsilon > 0$, we have

$$
A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : M(\frac{|x'_{ij} - L_1|}{\rho_1}) > \frac{\epsilon}{2}\} \in I, \tag{10.1}
$$

$$
A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : M(\frac{|y'_{ij} - L_2|}{\rho_2}) > \frac{\epsilon}{2}\} \in I. \tag{10.2}
$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex function, we have

$$M(\frac{|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\leq M(\frac{|\alpha||x'_{ij} - L_1|}{\rho_3}) + M(\frac{|\beta||y'_{ij} - L_2|}{\rho_3}).$$

$$\leq M(\frac{|x'_{ij} - L_1|}{\rho_1}) + M(\frac{|y'_{ij} - L_2|}{\rho_2})$$

Now, by [10.1] and [10.2],

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in_2 \mathcal{Z}^I(M)$. Hence ${}_2\mathcal{Z}^I(M)$ is a linear space.

Theorem 10.2.2. The spaces ${}_2m^I_{\mathcal{Z}}(M)$ and ${}_2m^I_{\mathcal{Z}_0}(M)$ are Banach spaces normed by

$$||x_{ij}|| = \inf\{\rho > 0 : \sup_{i,j} M(\frac{|x_{ij}|}{\rho}) \leq 1\}$$

Proof. Proof of this result is easy in view of the existing techniques and therefore is omitted.

Theorem 10.2.3. Let M_1 and M_2 be Orlicz functions that satisfy the $\triangle_2$-condition. Then
(a) $X(M_2) \subseteq X(M_1.M_2)$;
(b) $X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ For $X = {}_2\mathcal{Z}^I, {}_2\mathcal{Z}^I_0, {}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$.

Proof. (a) Let $(x_{ij}) \in {}_2\mathcal{Z}^I_0(M_2)$. Then there exists $\rho > 0$ such that

$$I - \lim_{i,j} M_2(\frac{|x'_{ij}|}{\rho}) = 0 \qquad [10.3]$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{ij} = M_2(\frac{|x'_{ij}|}{\rho})$ and consider for all $(i, j) \in \mathbb{N} \times \mathbb{N}$ we have

$$\lim_{0 \leq y_{ij} \leq \delta} M_1(y_{ij}) = \lim_{y_{ij} \leq \delta} M_1(y_{ij}) + \lim_{y_{ij} > \delta} M_1(y_{ij}).$$

We have

$$\lim_{y_{ij} \leq \delta} M_1(y_{ij}) \leq M_1(2) \lim_{y_{ij} \leq \delta} (y_{ij}). \qquad [10.4]$$

For $(y_{ij}) > \delta$, we have

$$(y_{ij}) < (\frac{y_{ij}}{\delta}) < 1 + (\frac{y_{ij}}{\delta}).$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(y_{ij}) < M_1(1 + (\frac{y_{ij}}{\delta})) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_{ij}}{\delta})$$

Since M_1 satisfies the $\triangle_2$-condition, we have

$$M_1(y_{ij}) < \frac{1}{2} K(\frac{y_{ij}}{\delta}) M_1(2) + \frac{1}{2} K(\frac{y_{ij}}{\delta}) M_1(2) = K(\frac{y_{ij}}{\delta}) M_1(2).$$

Hence

$$\lim_{y_{ij} > \delta} M_1(y_{ij}) \leq max(1, K\delta^{-1} M_1(2)) \lim_{y_{ij} > \delta} (y_{ij}). \qquad [10.5]$$

From [10.3], [10.4] and [10.5], we have $(x_{ij}) \in \mathcal{Z}_0^I(M_1).(M_2)$. Thus

$$\mathcal{Z}_0^I(M_2) \subseteq \mathcal{Z}_0^I(M_1.M_2).$$

The other cases can be proved similarly.

(b) Let $(x_k) \in \mathcal{Z}_0^I(M_1) \cap \mathcal{Z}_0^I(M_2)$. Then there exists $\rho > 0$ such that $I - \lim_k M_1(\frac{|x_k^/|}{\rho}) = 0$ and $I - \lim_k M_2(\frac{|x_k^/|}{\rho}) = 0$. The rest of the proof follows from the following equality

$$\lim_{k\in\mathbb{N}}(M_1 + M_2)(\frac{|x_k^/|}{\rho}) = \lim_{k\in\mathbb{N}} M_1(\frac{|x_k^/|}{\rho}) + \lim_{k\in\mathbb{N}} M_2(\frac{|x_k^/|}{\rho})$$

Theorem 10.2.4. The spaces ${}_2\mathcal{Z}_0^I(M)$ and ${}_2m_{\mathcal{Z}_0}^I(M)$ are solid and monotone .

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I(M)$. For $m_{\mathcal{Z}_0}^I(M)$ the result can be proved similarly. Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(M)$. Then there exists $\rho > 0$ such that

$$I - \lim_{i,j} M(\frac{|x_{ij}^{'}|}{\rho}) = 0 \qquad [10.6]$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [10.6] and the following inequality for all

$$M(\frac{|\alpha_{ij}x_{ij}^{'}|}{\rho}) \leq |\alpha_{ij}|M(\frac{|x_{ij}^{'}|}{\rho}) \leq M(\frac{|x_{ij}^{'}|}{\rho}).$$

By Lemma 1.12, a sequence space E is solid implies that E is monotone. We have the space ${}_2\mathcal{Z}_0^I(M)$ is monotone.

Theorem 10.2.5. The spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2m_{\mathcal{Z}}^I(M)$ are neither solid nor monotone in general.

Proof. Here we give a counter example. Let $I = I_\delta$ and $M(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(M)$ of $X(M)$ defined as follows, Let $(x_{ij}) \in X(M)$ and let $(y_{ij}) \in X_K(M)$ be such that

$$y_{ij} = \begin{cases} x_{ij}, & \text{if (i+j) is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence x_{ij} defined by $x_{ij} = 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$ but its K-stepspace preimage does not belong to ${}_2\mathcal{Z}^I(M)$. Thus ${}_2\mathcal{Z}^I(M)$ is not monotone.

Hence ${}_2\mathcal{Z}^I(M)$ is not solid.

Theorem 10.2.6. The spaces ${}_2\mathcal{Z}^I_0(M)$ and ${}_2\mathcal{Z}^I(M)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $M(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i+j} \quad and \;\; y_{ij} = i + j$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}^I_0(M)$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}^I_0(M)$. Hence the spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}^I_0(M)$ are not convergence free.

Theorem 10.2.7. The spaces ${}_2\mathcal{Z}^I_0(M)$ and ${}_2\mathcal{Z}^I(M)$ are sequence algebras.

Proof. We prove that ${}_2\mathcal{Z}^I_0(M)$ is a sequence algebra. For the space ${}_2\mathcal{Z}^I(M)$, the result can be proved similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I_0(M)$. Then

$$I - \lim M(\frac{|x'_{ij}|}{\rho_1}) = 0$$

and

$$I - \lim M(\frac{|y'_{ij}|}{\rho_2}) = 0$$

Let $\rho = \rho_1.\rho_2 > 0$. Then we can show that

$$I - \lim M(\frac{|(x'_{ij}.y'_{ij})|}{\rho}) = 0.$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}^I_0(M)$. Hence ${}_2\mathcal{Z}^I_0(M)$ is a sequence algebra.

Theorem 10.2.8. Let M be an Orlicz function. Then the inclusions

$$ {}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M) \subset {}_2\mathcal{Z}_\infty^I(M) $$

hold.

Proof: Let $(x_{ij}) \in {}_2\mathcal{Z}^I(M)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$ I - \lim M(\frac{|x_{ij}^{'} - L|}{\rho}) = 0. $$

We have $M(\frac{|x_{ij}^{'}|}{2\rho}) \leq \frac{1}{2}M(\frac{|x_{ij}^{'} - L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho})$. Taking supremum over (i,j) both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I(M)$. The inclusion ${}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M)$ is obvious.

Theorem 10.2.9. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I(M)$ and ${}_2\mathcal{Z}_0^I(M)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $M(x) = x$ for all $x = (x_{ij})$. If

$$ x_{ij} = \begin{cases} 1, & \text{for } i, j \in A, \\ 0, & otherwise. \end{cases} $$

Then

$$ (x_{ij}) \in {}_2\mathcal{Z}_0^I(M) \subset {}_2\mathcal{Z}^I(M), $$

by lemma 1.14. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$.

Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$ \pi(k) = \begin{cases} \phi(k), & \text{for} k \in K, \\ \psi(k), & otherwise. \end{cases} $$

is a permutation on $\mathbb{N}$, but $(x_{\pi(i)\pi(j)}) \notin {}_2\mathcal{Z}^I(M)$ and $(x_{\pi(i)\pi(j)}) \notin {}_2\mathcal{Z}_0^I(M)$. Hence ${}_2\mathcal{Z}_0^I(M)$ and ${}_2\mathcal{Z}^I(M)$ are not symmetric.